A sharp error estimate for numerical Fourier transform of band-limited functions based on windowed samples

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Abstract: W. Dickmeis and R.J. Nessel published their first version of a quantitative extension of the classical uniform boundedness principle in [*J. Approx. Theory* 31 (1981), 161–174]. It is a general approach to finding counterexamples that prove sharpness of error estimates. So far applications of this principle include error bounds for approximation processes, cubature rules, ordinary and partial differential equations, and reconstruction from samples. Here we discuss the error of discrete approximations of the Fourier transform based on windowed samples for band-limited functions. The results can be applied to the Hann- and Blackmann-Harris-window but also to window-functions that enable higher orders of convergence. We describe a class of such windows.

1   Introduction

Shannon sampling theorem states that (under reasonable conditions) one can compute values of the Fourier transform $f^\wedge(\omega) := \int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du$ on the basis of samples if the function f is band-limited, i.e. $f^\wedge(\omega) = 0$ for $|\omega| > \Omega$, Ω being a positive constant. Let $\Delta t > 0$ be so small that the Shannon-Nyquist condition $\Delta t \leq \frac{\pi}{\Omega}$ is valid. Then for all $\omega \in [-\Omega, \Omega]$ one has

$$f^\wedge(\omega) = \Delta t \sum_{k=-\infty}^{\infty} f(k\Delta t) \exp\left(-i\omega k\Delta t\right).$$

Also, the sampling theorem is used to reconstruct functions (signals) from samples $f(k\Delta t)$. There has been a lot of work regarding the reconstruction of signals under weaker assumptions than used in the sampling theorem, for a survey see [7]. Especially, other kernels than the sinc function can be applied. [21] deals with kernels defined by some window functions that we will discuss, too. But in contrast to reconstruction here we focus on the approximation of $f^\wedge(\omega)$.

In engineering applications only a finite number of samples out of a finite interval $[-R, R]$ is available. This is equivalent to dealing with a modified function $f \cdot 1_{[-R,R]}$ where $1_{[-R,R]}$ is the rectangle function with $1_{[-R,R]}(t) = 1$ for $t \in [-R, R]$ and $1_{[-R,R]}(t) = 0$ elsewhere. Unfortunately, this product no longer is band-limited unless it is the null function. Therefore, one has to cope with two errors: the difference between the transforms of f and $f \cdot 1_{[-R,R]}$ (truncation or leakage) and an approximation error that is

called aliasing error by engineers. This aliasing error originates from the absence of band-limitation of $f \cdot 1_{[-R,R]}$.

In the periodic case we analyzed the aliasing error without the influence of truncation for non-continuous functions in [14]. Since we now deal with band-limited functions f, it seems natural to require continuity of f so that f equals the continuous inverse Fourier transform of $f^{\wedge}$.

Let us introduce some notations. $L^1(\mathbb{R})$ is the space of absolutely integrable (complex-valued) functions on $\mathbb{R}$, the set of real numbers, with norm $\|f\|_1 := \int_{\mathbb{R}} |f(t)| \, dt$. Also, we use

$$\|f\|_{1,[a,b]} := \int_a^b |f(t)| \, dt, \text{ and } \|f\|_{\infty,[a,b]} := \sup_{t \in [a,b]} |f(t)|, \|f\|_\infty := \|f\|_{\infty,\mathbb{R}}.$$

To measure smoothness of functions we work with the well-known moduli of continuity. Let $n \in \mathbb{N}$ where $\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of natural numbers. The n-th difference of a function f at point t is defined as

$$\Delta_h^1 f(t) := f(t+h) - f(t), \quad \Delta_h^n f(t) := \Delta_h^1 \Delta_h^{n-1} f(t), \, n > 1, \text{ or}$$

$$\Delta_h^n f(t) := \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(t+jh),$$

and moduli of continuity are defined via n-th differences:

$$\omega_n(f, \delta, C(\mathbb{R})) := \sup_{0 < h < \delta} \|\Delta_h^n f(\cdot)\|_\infty, \quad \omega_n(f, \delta, L^1(\mathbb{R})) := \sup_{0 < h < \delta} \|\Delta_h^n f(\cdot)\|_1.$$

To reduce truncation and aliasing errors, often "better" window functions than $1_{[-R,R]}$ are used (see [15]). Here we investigate window functions g_R with compact support in $[-R, R]$ that comply with following requirements:

(W1) $g_R(t) = g\left(\frac{t}{R}\right)$ for an even function g with compact support in $[-1, 1]$.

(W2) $g(0) = 1$.

(W3) g is $r+1$-times differentiable on $\mathbb{R}$ for an even $r = 2s$, $s \in \mathbb{N} \cup \{0\}$, and

$$\omega_2(\delta, g^{(r+1)}, L^1(\mathbb{R})) = \mathcal{O}(\delta^2), \quad \delta \to 0+.$$

For example, this asymptotic behaviour is given if $g^{(r+1)}$ is continuous and piecewise composed of a finite number of two-times continuously differentiable functions.

(W4) First r moments of $g^{\wedge}$ are zero: $\int_{-\infty}^\infty u^k g^{\wedge}(u) \, du = 0$ for each $1 \le k \le r$ (for odd k this obviously is fulfilled because g is even; (W4) is an empty condition for $r = 0$).

(W5) $|1 - g_R(t)| \le C_t R^{-r-2}$ (e.g. this is fulfilled if g is $r+2$-times differentiable in a neighbourhood of 0 with $g^{(k)}(0) = 0$, $1 \le k \le r+1$, i.e. g approximates $1_{[-1,1]}$ near $t = 0$).

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Classical window functions that meet (W1–W5) for $r = 0$ are Hann-window (also known as $\cos^2$-, von Hann-, and Hanning-window), Blackmann- and Blackmann-Harris-window functions. They are special cases of conventional windows in the form of a sum of cosine terms (cf. (W1), $m \in \mathbb{N}$, $r = 2s$, $s \in \mathbb{N} \cup \{0\}$)

$$g(t) := \begin{cases} \sum_{k=0}^{m} a_k \cos(\pi k t) & : \quad -1 \le t \le 1 \\ 0 & : \quad |t| > 1, \end{cases} \tag{1.1}$$

where the constants a_k have to be chosen such that (for (1.2) cf. [19], for (1.3) see [15, p. 63])

$$\sum_{k=0}^{m} (-1)^k a_k = 0, \quad \sum_{k=0}^{m} (-1)^k k^{2j} a_k = 0 \quad \text{for all } 1 \le j \le s, \tag{1.2}$$

$$\sum_{k=0}^{m} a_k = 1, \tag{1.3}$$

$$\sum_{k=0}^{m} k^{2j} a_k = 0 \quad \text{for all } 1 \le j \le s. \tag{1.4}$$

Equation (1.3) implies (W2): $g(0) = \sum_{k=0}^{m} a_k = 1$.

Condition (W3) holds true, because (1.2) ensures that all derivatives up to the order $r + 1$ exist (especially at ± 1). $g^{(r+1)}$ is continuous and infinitely often differentiable on $[-1, 1]$, $(-\infty, -1]$, and $[1, \infty)$.

Condition (W5) is fulfilled, too: Let $R > |t|$, then via Taylor-expansion of cosine and (1.3, 1.4) we find some real numbers $\xi_{\frac{\pi k t}{R}}$ such that $(R \to \infty)$

$$\begin{aligned}
|1 - g_R(t)| &= \left| 1 - \sum_{k=0}^{m} a_k \cos\left(\frac{\pi k t}{R}\right) \right| \\
&= \left| 1 - \sum_{k=0}^{m} a_k \left[\sum_{j=0}^{s} (-1)^j \frac{\left(\frac{\pi k t}{R}\right)^{2j}}{(2j)!} + (-1)^{s+1} \frac{\left(\frac{\pi k t}{R}\right)^{2s+2}}{(2s)!} \cos\left(\xi_{\frac{\pi k t}{R}}\right) \right] \right| \\
&= \left| 1 - \left[\sum_{j=0}^{s} (-1)^j \frac{\left(\frac{\pi t}{R}\right)^{2j}}{(2j)!} \sum_{k=0}^{m} k^{2j} a_k \right] - \sum_{k=0}^{m} (-1)^{s+1} \frac{\left(\frac{\pi k t}{R}\right)^{2s+2}}{(2s)!} \cos\left(\xi_{\frac{\pi k t}{R}}\right) \right| \\
&= \left| \sum_{k=0}^{m} a_k \frac{\left(\frac{\pi k t}{R}\right)^{2s+2}}{(2s)!} \cos\left(\xi_{\frac{\pi k t}{R}}\right) \right| = \mathcal{O}_t(R^{-r-2}).
\end{aligned}$$

With condition (W4) for $r > 0$ we deal in section 3.

Depending on m and a_k we get following windows for $r = 2s = 0$ (see [15]):

- For $m = 1$ and $a_0 = a_1 = \frac{1}{2}$ the function g_R is the Hann-window function:

$$g_R(t) = \begin{cases} \frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi}{R} t\right) = \cos^2\left(\frac{\pi}{2R} t\right) & : \quad -R \le t \le R \\ 0 & : \quad |t| > R. \end{cases}$$

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- For $m = 2$ and $a_0 = \frac{1-c}{2}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{c}{2}$, $0 < c < 1$, the function g_R is called Blackmann-window.

- The case $m = 3$ for (rounded) constants $a_0 = 0.359$, $a_1 = 0.488$, $a_2 = 0.141$, and $a_3 = 0.012$ is the Blackmann-Harris-window.

In engineering applications often a fixed R is used, whereas we discuss $R \to \infty$. For a fixed R it turns out that violating (1.2) might give better results. In this context the settings for the Hamming-window are $m = 1$, $a_0 = 0.54$, $a_1 = 0.46$ (cf. [23] for window design).

In this paper we investigate the error of replacing $(f^\wedge)^*(\omega)$ by $([f \cdot g_R]^\wedge)^*(\omega)$ where $(f^\wedge)^*(\omega)$ denotes a value of the discrete Fourier transform

$$(f^\wedge)^*(\omega) := \Delta t \sum_{k \in \mathbb{Z},\, k\Delta t \in [-R,R]} f(k\Delta t) \exp(-i\omega k \Delta t), \tag{1.5}$$

where $\mathbb{Z} := \{0, 1, -1, 2, -2, \dots\}$.

The rate of convergence is determined by (W3, W4) and the smoothness of $f^\wedge$.

Theorem 1.1 *Let $f \in L^1(\mathbb{R})$ be a continuous, band-limited function ($f^\wedge(\omega) = 0$ outside $[-\Omega, \Omega]$). Further, $f^\wedge$ should be smooth in the sense that $(f^\wedge)^{(r)}$ exists for an even $r = 2s$, $s \in \mathbb{N} \cup \{0\}$, and*

$$\omega_2((f^\wedge)^{(r)}, \delta, C(\mathbb{R})) \le C\delta^\alpha$$

for some $0 < \alpha < 2$. Let $\Delta t > 0$ fulfill the Shannon-Nyquist condition

$$\frac{\pi}{\Delta t} \ge \Omega. \tag{1.6}$$

The window function $g_R(t) = g(t/R)$ might fulfill (W1–W4).

Then for each $0 < \Omega_1 < \Omega$ the following direct estimate holds true ($R \to \infty$):

$$\sup_{\omega \in [-\Omega_1, \Omega_1]} |f^\wedge(\omega) - [(f \cdot g_R)^\wedge]^*(\omega)| = \mathcal{O}(R^{-r-\alpha}). \tag{1.7}$$

Smoothness of $f^\wedge$ (that is measured via the ω_2-modulus of the r-th derivative) is closely connected to the behavior of the tail integral of f (cf. [1]).

Estimate (1.7) is best possible in the following sense.

Theorem 1.2 *Let $|u_1| \le \Omega_1 < \Omega$ and $\Delta t > 0$ be constants such that (1.6) is fulfilled. Let $r = 2s$, $s \in \mathbb{N} \cup \{0\}$, and let the window function $g_R(t) = g(t/R)$ fulfill (W1–W3).*

Then for each $0 < \alpha < 2$ with $r + \alpha > 1$ there exists a continuous, real-valued counterexample $f_\alpha \in L^1(\mathbb{R})$ that is band-limited in the sense of $f^\wedge(\omega) = 0$ for all $|\omega| > \Omega$, such that $(f^\wedge)^{(r)}$ exists and

$$\omega_2((f_\alpha^\wedge)^{(r)}, \delta, C(\mathbb{R})) \le C\delta^\alpha$$

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but $(R \to \infty)$

$$|f_\alpha^\wedge(u_1) - [(f_\alpha \cdot g_R)^\wedge]^*(u_1)| \neq o(R^{-r-\alpha}). \qquad (1.8)$$

If additionally (W5) holds true, then counterexample f_α does not only behave like (1.8) at the point u_1 but for all $\omega \in [-\Omega_1, \Omega_1]$ there simultaneously is

$$|f_\alpha^\wedge(\omega) - [(f_\alpha \cdot g_R)^\wedge]^*(\omega)| \neq o(R^{-r-\alpha}).$$

For window functions of type (1.1) fulfilling (1.2, 1.3) like the Hann- or Blackmann-Harris-window, Theorem 1.1 holds true for $r = 0$, $0 < \alpha < 2$, and the estimate is best possible at least for $1 < \alpha < 2$ simultaneously on a set $[-\Omega_1, \Omega_1]$. Examples of window functions fulfilling (1.2–1.4) and (W1–W5) for $r \geq 2$ are given in section 3. For these examples Theorem 1.1 provides an estimate for $r \geq 2$ that is best possible for all values $0 < \alpha < 2$ on a set $[-\Omega_1, \Omega_1]$.

In what follows we discuss properties of window functions in the frequency domain, give examples for convergence of order $r > 0$ and prove Theorem 1.1. In order to estimate the error, we split it up into a truncation and an aliasing part:

$$|f^\wedge(\omega) - [(f \cdot g_R)^\wedge]^*(\omega)| \leq \underbrace{|f^\wedge(\omega) - (f \cdot g_R)^\wedge(\omega)|}_{\text{truncation error}} + \underbrace{|(f \cdot g_R)^\wedge(\omega) - [(f \cdot g_R)^\wedge]^*(\omega)|}_{\text{aliasing error}}.$$

We discuss each error in a separate section and conclude with the proof of Theorem 1.2.

Proofs of Theorems 1.1 and 1.2 discuss properties of window functions in terms of Approximation Theory. This perspective might also be helpful to engineers working in the field of signal analysis.

2   Window functions as kernels in frequency domain

Window functions that fulfill (W1–W3) can be interpreted as kernels of Fejér-type in the frequency domain.

We define an even (cf. (W1)), real-valued, continuous kernel χ via (inverse) Fourier transform:

$$\chi(u) := \frac{1}{2\pi} g^\wedge(u) = \frac{1}{2\pi} \int_{-1}^1 g(v) e^{-ivu} \, dv.$$

Well-known Riemann-Lebesgue-lemma with orders (cf. [22]) gives the estimate

$$|\chi(u)| = \frac{1}{2\pi} |g^\wedge(u)| \leq C \omega_{r+3} \left(\frac{\pi}{|u|}, g, L^1(\mathbb{R}) \right).$$

As a consequence of (W3) we get

$$\omega_{r+3} \left(\frac{\pi}{|u|}, g, L^1(\mathbb{R}) \right) \leq C_1 |u|^{-r-1} \omega_2 \left(\frac{\pi}{|u|}, g^{(r+1)}, L^1(\mathbb{R}) \right) \leq C_2 |u|^{-r-3}.$$

We have shown that

$$|\chi(u)| \leq C|u|^{-r-3}. \tag{2.1}$$

That means that $\chi \in L^1(\mathbb{R})$, χ is Fourier transformable with $\chi^\wedge(t) = g(t)$ so that χ is band-limited. The sidelobe falloff rate of order $r+3$ implies that the $(r+\alpha)$-th absolute moment of χ exists for $0 < \alpha < 2$:

$$m(\chi, r + \alpha) := \int_{-\infty}^{\infty} |u|^{r+\alpha} |\chi(u)| \, du < \infty. \tag{2.2}$$

Because of (W2) the kernel is normed:

$$\int_{-\infty}^{\infty} \chi(u) \, du = \chi^\wedge(0) = g(0) = 1. \tag{2.3}$$

We can write $g_R(t)$ with χ as follows:

$$[R\chi(R\cdot)]^\wedge(t) = \chi^\wedge\left(\frac{t}{R}\right) = g_R(t). \tag{2.4}$$

The set of functions $\{R\chi(Rx) : R > 0\}$ (and also the function χ) is called a kernel of Fejér-type. In Approximation Theory it serves as an approximate identity (cf. [6, p. 121]). In section 5 we estimate the truncation error as a remainder of such an approximation process.

Condition (W4) is a requirement for moments of the kernel that we verify for a certain class of window-functions in section 3:

$$0 = \int_{-\infty}^{\infty} u^k g^\wedge(u) \, du = 2\pi \int_{-\infty}^{\infty} u^k \chi(u) \, du. \tag{2.5}$$

The continuous kernel corresponding to (1.1) is a well-known linear combination of sinc-functions where $\mathrm{sinc}(u) = \sin(u)/u$. By partial integration we get

$$\begin{aligned}
\chi(u) &= -\frac{u \sin(u)}{\pi} \sum_{k=0}^{m} \frac{a_k(-1)^k}{k^2\pi^2 - u^2} = \frac{\sin(u)}{2\pi} \sum_{k=0}^{m} a_k(-1)^k \left[\frac{1}{k\pi + u} - \frac{1}{k\pi - u}\right] \\
&= \frac{1}{2\pi} \sum_{k=0}^{m} a_k \left[\mathrm{sinc}(k\pi + u) + \mathrm{sinc}(k\pi - u)\right],
\end{aligned} \tag{2.6}$$

especially the Hann-window function leads to the kernel $\chi(u) = \frac{1}{2\pi} \frac{\mathrm{sinc}(u)}{1 - \frac{u^2}{\pi^2}}$.

3 Higher order of convergence

So far our examples are chosen for $r = 0$. To get window-functions for higher orders $r = 2s$, $s \in \mathbb{N}$, we restrict ourselves to the case where $m = r + 1$.

Please note that in this case there is a unique solution to linear equations (1.2–1.4): With Gauss operations we transform matrix A representing equations (1.3), (1.4), and (1.2) into the matrix B:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
0 & 1 & 4^1 & 9^1 & 16^1 & 25^1 & \\
0 & 1 & 4^2 & 9^2 & 16^2 & 25^2 & \\
\vdots & & & & & & \\
1 & -1 & 1 & -1 & 1 & -1 & \cdots \\
0 & -1 & 4^1 & -9^1 & 16^1 & -25^1 & \\
0 & -1 & 4^2 & -9^2 & 16^2 & -25^2 & \\
\vdots & & & & & &
\end{bmatrix}
,
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 4^1 & 0 & 16^1 & 0 & \\
0 & 0 & 4^2 & 0 & 16^2 & 0 & \\
\vdots & & & & & & \\
0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
0 & 1 & 0 & 9^1 & 0 & 25^1 & \\
0 & 1 & 0 & 9^2 & 0 & 25^2 & \\
\vdots & & & & & &
\end{bmatrix}
.
$$

$$\underbrace{}_{A} \qquad \underbrace{}_{B}$$

First $\frac{r}{2}+1$ rows of B without columns of zeroes make up a Vandermode matrix. The same is true for the remaining $\frac{r}{2}+1$ rows. Therefore, both groups of equations are linear independent. Because the groups have different zero columns, it immediately follows that all rows in B (and therefore in A) are linear independent such that there is a unique solution.

The next observation is that $a_k = 0$, $k \in \{2, 4, \ldots, r\}$. To see this, we combine homogeneous equations (1.2) and (1.4) for $1 \leq j \leq s$ by adding them in pairs. Then we get s linear independent equations for the s variables $a_2, a_4, \ldots, a_r$ that are of Vandermonde-type as well (cf. rows 2 to $\frac{r}{2}+1$ in B). As a solution of the homogeneous system the variables are zero.

We show that with these parameters (W4) is fulfilled. To this end, we compute moments (2.5) for $k = 2j$, $1 \leq j \leq s = \frac{r}{2}$. Please note that the highest power of the nominator is less than the highest power of the denominator in the fraction part of the following integral. The reason is tail estimate (2.1). Therefore we can use partial fraction decomposition with constants A_k. With $J := \{1, 3, 5, \ldots r + 1\}$ we get (cf. (2.6))

$$
\int_{-\infty}^{\infty} u^{2j} g^{\wedge}(u)\, du = 4\pi \int_0^{\infty} u^{2j} \chi(u)\, du = \int_0^{\infty} \sin(u) \sum_{k \in J} \left[\frac{A_k}{u - k\pi} + \frac{A_k}{u + k\pi} \right] du \qquad (3.1)
$$

$$
= \sum_{k \in J} A_k \int_0^{\infty} \frac{\sin(u)}{u - k\pi} + \frac{\sin(u)}{u + k\pi}\, du
$$

$$
= \sum_{k \in J} A_k (-1)^k \int_0^{\infty} \frac{\sin(u - k\pi)}{u - k\pi} + \frac{\sin(u + k\pi)}{u + k\pi}\, du
$$

$$
= \sum_{k \in J} A_k (-1)^k \left[\frac{\pi}{2} + \mathrm{Si}(k\pi) + \frac{\pi}{2} - \mathrm{Si}(k\pi) \right] = \pi \sum_{k \in J} A_k (-1)^k = -\pi \sum_{k \in J} A_k = 0.
$$

Note that terms $\frac{A_k}{u - k\pi}$ and $\frac{A_k}{u + k\pi}$ share the same constant A_k. In the last step we use that $\sum_{k \in J} 2A_k$ is the coefficient of $u^{2(r+1-r/2)-1} = u^{r+1}$ in the nominator if we write the sum in (3.1) as one fraction. The highest power of the denominator then is $2(r + 1 - r/2) =$

$r + 2$. Because of (2.1) there can be no higher power than $2j + (r + 2) - (r + 3) \leq r - 1$ in the nominator. Therefore $\sum_{k \in J} A_k = 0$.

We have proved that for each $r = 2s$, $s \in \mathbb{N}$, there is a window-function fulfilling (W1–W5) so that error estimate Theorem 1.1 gives a sharp error bound of order $\mathcal{O}(R^{-r-\alpha})$, $0 < \alpha < 2$.

Coefficients in the case $r = 4$, $m = 3$ are $a_0 = \frac{1}{2}$, $a_1 = \frac{9}{16}$, $a_2 = 0$, and $a_3 = -\frac{1}{16}$. They define the window-function

$$g_R(t) = \begin{cases} \frac{1}{2} + \frac{9}{16} \cos\left(\frac{\pi}{R}t\right) - \frac{1}{16} \cos\left(\frac{3\pi}{R}t\right) & : \quad -R \leq t \leq R \\ 0 & : \quad |t| > R. \end{cases}$$

This window was introduced in [2] for aesthetic reasons but not as a result of mathematical optimiziation (as it is said in [2]).

4   Aliasing error

It is well known that the Shannon sampling theorem can be proved via the Poisson summation formula. We use this approach to estimate the aliasing error.

Let f, $\|f\|_1 < \infty$, be band-limited to $[-\Omega, \Omega]$, and $g_R(t) = g(t/R)$ be a window function satisfying (W1–W3).

The inverse Fourier transform $(f^\wedge)^\vee$ of $f^\wedge$ does exist because of band limitation. In Theorem 1.1 we investigate a continuous function f. For the next arguments the continuity is not required. Without continuity there is $f = (f^\wedge)^\vee$ a.e.

$(f \cdot g_R)^\wedge(\omega)$ can be written as the convolution

$$\frac{1}{2\pi}(f^\wedge * g_R^\wedge)(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(u) g_R^\wedge(\omega - u)\, du = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} f^\wedge(u) g_R^\wedge(\omega - u)\, du.$$

We apply Poisson summation to the function

$$h(\omega) := \Delta t (f(\cdot \Delta t) g_R(\cdot \Delta t))^\wedge(\omega) = (f g_R)^\wedge \left(\frac{\omega}{\Delta t}\right) = \frac{1}{2\pi}[f^\wedge * g_R^\wedge]\left(\frac{\omega}{\Delta t}\right)$$

where parameter $\Delta t > 0$ is fixed.

We verify preliminaries of the Poisson summation formula (cf. [6, p. 202]):

- As a Fourier transform, h is continuous on $\mathbb{R}$.

- The function h is absolutely integrable because

$$\int_{-\infty}^{\infty} |h(\omega)|\, d\omega \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f^\wedge\left(\frac{\omega}{\Delta t} - u\right) \right| |g_R^\wedge(u)|\, d\omega du$$

$$= \frac{\Delta t}{2\pi} \|g_R^\wedge\|_1 \int_{-\Omega}^{\Omega} |f^\wedge(\omega)| d\omega < \infty.$$

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- Sequence $(h^\wedge(k))_{k\in\mathbb{Z}}$ is absolutely summable:

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} |h^\wedge(k)| &= 2\pi\Delta t \sum_{k=-\infty}^{\infty} |([f^\wedge]^\vee g_R)(-k\Delta t)| \\
&= 2\pi\Delta t \sum_{k\in\mathbb{Z}\cap[-R/\Delta t, R/\Delta t]} |([f^\wedge]^\vee g_R)(k\Delta t)| < \infty.
\end{aligned}
$$

- We show that the series $\sum_{k=-\infty}^{\infty} h(\omega + k2\pi)$ converges uniformly in ω on $[0, 2\pi]$ (and therefore on $\mathbb{R}$). To this end, we estimate the summands:

$$
\begin{aligned}
|h(\omega + k2\pi)| &= \frac{1}{2\pi}\left| \int_{-\Omega}^{\Omega} f^\wedge(u) g_R^\wedge\left(\frac{\omega}{\Delta t} + k\frac{2\pi}{\Delta t} - u\right) du \right| \\
&\leq \frac{1}{2\pi}\|f^\wedge\|_{\infty,[-\Omega,\Omega]}\|g_R^\wedge\|_{1,[-\Omega+k2\pi/\Delta t, \Omega+(k+1)2\pi/\Delta t]}. \qquad (4.1)
\end{aligned}
$$

The right side is independent of ω. Let $l \in \mathbb{N}$ such that $\Omega \leq l\frac{2\pi}{\Delta t}$. Then

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} &\|g_R^\wedge\|_{1,[-\Omega+k2\pi/\Delta t, \Omega+(k+1)2\pi/\Delta t]} \\
&\leq \sum_{k=-\infty}^{\infty} \|g_R^\wedge\|_{1,[(-l+k)2\pi/\Delta t,(l+k+1)2\pi/\Delta t]} = (2l+1)\|g_R^\wedge\|_1.
\end{aligned}
$$

The convergent majorant (4.1) proves uniform convergence.

Now we can apply the Poisson summation formula: For each $\omega_0 \in \mathbb{R}$ there holds true

$$
\sum_{k=-\infty}^{\infty} h(\omega_0 + k2\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h^\wedge(k)e^{ik\omega_0},
$$

i.e.

$$
\begin{aligned}
\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} [f^\wedge * g_R^\wedge]\left(\frac{\omega_0}{\Delta t} + k\frac{2\pi}{\Delta t}\right) &= \Delta t \sum_{k=-\infty}^{\infty} [f^\wedge]^\vee(-k\Delta t)g_R(-k\Delta t)e^{ik\omega_0} \\
&= \Delta t \sum_{k=-\infty}^{\infty} [f^\wedge]^\vee(k\Delta t)g_R(k\Delta t)e^{-ik\omega_0}.
\end{aligned}
$$

By setting $\omega_0 := \omega\Delta t$ we get

$$
\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} [f^\wedge * g_R^\wedge]\left(\omega + k\frac{2\pi}{\Delta t}\right) = \Delta t \sum_{k=-\infty}^{\infty} [f^\wedge]^\vee(k\Delta t)g_R(k\Delta t)e^{-i\omega k\Delta t}.
$$

This gives a formula for the aliasing error for continuous f (i.e. $(f^\wedge)^\vee = f$):

$$
(f \cdot g_R)^\wedge(\omega) - [(f \cdot g_R)^\wedge]^*(\omega)
$$

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$$= \frac{1}{2\pi}[f^{\wedge} * g_R^{\wedge}](\omega) - \Delta t \sum_{k=-\infty}^{\infty} f(k\Delta t)g_R(k\Delta t)e^{-i\omega k\Delta t}$$

$$= -\frac{1}{2\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} [f^{\wedge} * g_R^{\wedge}]\left(\omega + k\frac{2\pi}{\Delta t}\right) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} [f \cdot g_R]^{\wedge}\left(\omega + k\frac{2\pi}{\Delta t}\right).$$

With this formula we now prove an error estimate that is independent of the smoothness of $f^{\wedge}$. It only takes asymptotic behavior of kernel χ (cf. (2.4)) into account. At this point condition (1.6) is needed, i.e. $\Omega \leq \frac{\pi}{\Delta t}$. For each $\omega \in [-\Omega_1, \Omega_1]$ we get

$$|(f \cdot g_R)^{\wedge}(\omega) - [(f \cdot g_R)^{\wedge}]^*(\omega)|$$

$$\leq \frac{1}{2\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left| \int_{-\Omega}^{\Omega} f^{\wedge}(u)g_R^{\wedge}\left(\omega + k\frac{2\pi}{\Delta t} - u\right) du \right|$$

$$\leq \frac{1}{2\pi}\|f^{\wedge}\|_{\infty,[-\Omega,\Omega]} \sum_{k\in\mathbb{Z}\setminus\{0\}} \|g_R^{\wedge}\|_{1,\left[\omega+(2k-1)\frac{\pi}{\Delta t},\omega+(2k+1)\frac{\pi}{\Delta t}\right]}$$

$$= \frac{1}{2\pi}\|f^{\wedge}\|_{\infty,[-\Omega,\Omega]}\|g_R^{\wedge}\|_{1,\mathbb{R}\setminus\left[\omega-\frac{\pi}{\Delta t},\omega+\frac{\pi}{\Delta t}\right]} \leq \frac{1}{2\pi}\|f^{\wedge}\|_{\infty,[-\Omega,\Omega]}\|g_R^{\wedge}\|_{1,\mathbb{R}\setminus[\omega-\Omega,\omega+\Omega]}$$

$$= \frac{1}{2\pi}\|f^{\wedge}\|_{\infty,[-\Omega,\Omega]}\|Rg^{\wedge}(R\cdot)\|_{1,\mathbb{R}\setminus[\omega-\Omega,\omega+\Omega]} = \|f^{\wedge}\|_{\infty,[-\Omega,\Omega]}\|\chi\|_{1,\mathbb{R}\setminus[R(\omega-\Omega),R(\omega+\Omega)]}.$$

Please note that $\omega + \Omega \geq \Omega - \Omega_1 > 0$, and $\omega - \Omega \leq \Omega_1 - \Omega < 0$. Together with (2.1) we conclude

$$\|\chi\|_{1,[R(\omega+\Omega),\infty)} \leq \|\chi\|_{1,[R(\Omega-\Omega_1),\infty)} \leq \frac{C}{r+2}(\Omega - \Omega_1)^{-r-2}R^{-r-2},$$

$$\|\chi\|_{1,(-\infty,R(\omega-\Omega)]} \leq \|\chi\|_{1,(-\infty,R(\Omega_1-\Omega)]} \leq \frac{C}{r+2}(\Omega - \Omega_1)^{-r-2}R^{-r-2},$$

i.e., $|(f \cdot g_R)^{\wedge}(\omega) - [(f \cdot g_R)^{\wedge}]^*(\omega)| = \mathcal{O}(R^{-r-2})$ independently of $\omega \in [-\Omega_1, \Omega_1]$. Because $r + \alpha < r + 2$ this proves that in the context of Theorem 1.1 the aliasing error is of order $\mathcal{O}(R^{-r-\alpha})$.

5 Truncation error

Let $\chi \in L^1(\mathbb{R})$ be an even kernel that fulfills (2.2, 2.3, 2.5) for an $r = 2s$, $s \in \mathbb{N} \cup \{0\}$, i.e. the $(r + \alpha)$-th absolute moments of the normed kernel exist for $0 < \alpha < 2$ and all moments up to the r-th moment are zero. Then for every r-times continuously differentiable $\varphi : \mathbb{R} \to \mathbb{C}$ with

$$\omega_2(\varphi^{(r)}, \delta, C(\mathbb{R})) \leq C\delta^{\alpha}$$

there holds true (see [6, p.144])

$$\sup_{\omega \in \mathbb{R}} |\varphi(\omega) - [\varphi * R\chi(\cdot R)](\omega)| \leq CR^{-r-\alpha}.$$

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This classic estimate for a convolution process can be applied to estimate the truncation error under the preliminaries of Theorem 1.1 with $\varphi = f^\wedge$ and $R\chi(\cdot R) = \frac{1}{2\pi} g_R^\wedge$. Since conditions (2.2, 2.3, 2.5) follow from (W1–W4) (see section 2) we have shown

$$\sup_{\omega \in \mathbb{R}} |f^\wedge(\omega) - (f \cdot g_R)^\wedge(\omega)| \leq C R^{-r-\alpha}.$$

In connection with the result of the previous section this completes the proof of Theorem 1.1.

6 Sharpness

In this section we prove Theorem 1.2. Preliminaries ensure that the aliasing error vanishes with order $\mathcal{O}(R^{-r-2})$. Because $\alpha < 2$, it is sufficient to construct a counterexample f_α so that the truncation error is $\neq o(R^{-r-\alpha})$. This is the error of a convolution process of Fejér-type as discussed in the previous section. In [18] sharpness of such error bounds is shown on the basis of quantitative extensions of the uniform boundedness principle developed by Dickmeis, Nessel and van Wickeren (cf. [9, 10]). We follow this approach and modify it to fit for band-limited functions. In this manner we get counterexamples in the frequency domain. By inverse Fourier transform we then find the results of Theorem 1.2. Similar applications of the uniform boundedness principle on other fields of Numerical Analysis are presented in [5, 4, 3, 11, 12, 13]. In [20] it is applied to reconstruction from samples.

An abstract modulus of continuity is a function ω, continuous on $[0, \infty)$ such that, for $0 < \delta_1, \delta_2$,

$$0 = \omega(0) < \omega(\delta_1) \leq \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2). \tag{6.1}$$

Functions $\omega(\delta) := \delta^\beta$, $0 < \beta \leq 1$, satisfy these conditions.

For a Banach space X with norm $\| \cdot \|_X$ let $X^\sim$ be the set of non-negative-valued sublinear bounded functionals T on X, i.e., T maps X into $\mathbb{R}$ such that for all $f, g \in X$, $c \in \mathbb{R}$

$$Tf \geq 0, \qquad T(f + g) \leq Tf + Tg, \qquad T(cf) = |c| Tf,$$

$$\|T\|_{X^\sim} := \sup\{Tf : \|f\|_X \leq 1\} < \infty.$$

Theorem 6.1 *Suppose that for a family of remainders $\{T_{n,u} : n \in \mathbb{N}, u \in \mathbb{B}\} \subset X^\sim$, $\mathbb{B}$ being a non-empty index set, and for a measure of smoothness $\{S_\delta : \delta > 0\} \subset X^\sim$ there are test elements $h_n \in X$ and a constant $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $n \in \mathbb{N}$, and all $0 < \delta \ (\leq 1)$:*

$$\|h_n\|_X \quad \leq \quad C_1, \tag{6.2}$$

$$S_\delta h_n \quad \leq \quad C_2 \min\left\{1, \frac{\sigma(\delta)}{\varphi_n}\right\}, \tag{6.3}$$

$$T_{n,u_1} h_n \quad \geq \quad C_{3,u_1} > 0, \tag{6.4}$$

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where $\sigma(\delta)$ is a function, strictly positive on $(0, \infty)$, and $(\varphi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ is a strictly decreasing sequence with $\lim_{n \to \infty} \varphi_n = 0$, and $u_1 \in \mathbb{B}$ is a fixed index. Then for each modulus ω satisfying (6.1) and

$$\lim_{\delta \to 0+} \frac{\omega(\delta)}{\delta} = \infty$$

there exists a counterexample $f_\omega \in X$,

$$f_\omega = \sum_{k=1}^{\infty} \omega\left(\varphi_{n_k}\right) h_{n_k} \tag{6.5}$$

for a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $(\delta \to 0+, n \to \infty)$

$$\begin{aligned} S_\delta f_\omega &= \mathcal{O}\left(\omega(\sigma(\delta))\right), \\ T_{n,u_1} f_\omega &\neq o(\omega(\varphi_n)). \end{aligned}$$

The sequence $(n_k)_{k=1}^{\infty}$ can be chosen such that

$$T_{n,u} f_\omega \neq o(\omega(\varphi_n)) \tag{6.6}$$

simultaneously for all $u \in \mathbb{B}$, if instead of (6.4) following conditions are satisfied for all $u \in \mathbb{B}$ and $1 \leq j \leq n-1$, $n > n_0$:

$$\begin{aligned} \|T_{n,u}\|_{X^\sim} &\leq C_{4,n}, \tag{6.7} \\ T_{n,u} h_j &\leq C_{5,u} C_{5,j} \varphi_n, \tag{6.8} \\ T_{n,u} h_n &\geq C_{6,u} > 0. \tag{6.9} \end{aligned}$$

For a constructive proof of the first part using a gliding hump, further comments, and applications to Approximation Theory see [10]. The extension to the index set $\mathbb{B}$ is a special case of more general theorems proven in [9, 16, 17].

We use this general concept to show Theorem 1.2. To this end let $\sigma(\delta) := \delta^2$, $\varphi_n := \frac{1}{n^2}$, and $\omega(\delta) := \delta^{\alpha/2}$. Let X be the space of r-times continuously differentiable functions with compact support in $[-\Omega, \Omega]$ equipped with sup-norm $\|f\|_{r,\infty} := \sum_{k=0}^{r} \|f^{(r)}\|_\infty$. For $f \in X$ we set $S_\delta f := \omega_2(f^{(r)}, \delta, C(\mathbb{R}))$.

For construction of test elements we place a constant Ω_0 between Ω_1 and Ω: $0 < |u_1| \leq \Omega_1 < \Omega_0 < \Omega$.

Functionals $T_n := T_{n,u}$ express the truncation error at the point u (especially for $u = u_1$):

$$T_{n,u} f := n^r |f(u) - [f * n\chi(\cdot n)](u)|.$$

Therefore, $T_{n,u} \in X^\sim$ with $\|T_{n,u}\|_{X^\sim} \leq n^r [1 + \|\chi\|_1]$.

The sequence of test elements is constructed from functions

$$\tilde{h}_n(u) := \frac{1}{n^r} \exp(iu\omega_0 n).$$

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Here, $\omega_0 \in \mathbb{R}$ is a constant such that $|\chi^\wedge(\omega_0)| \leq \frac{1}{2}$ (note that $\lim_{u\to\pm\infty} \chi^\wedge(u) = 0$). These functions ensure that

$$
\begin{aligned}
T_{n,u}\tilde{h}_n &= n^r|\tilde{h}_n(u) - [\tilde{h}_n * n\chi(\cdot n)](u)| = \left| e^{iu\omega_0 n} - e^{iu\omega_0 n} n \int_{-\infty}^{\infty} e^{-i\omega_0 nt}\chi(nt)\, dt \right| \\
&= \left| 1 - \int_{-\infty}^{\infty} e^{-i\omega_0 v}\chi(v)\, dv \right| = |1 - \chi^\wedge(\omega_0)| \geq \frac{1}{2}.
\end{aligned}
$$

Functions $\tilde{h}_n$ do not have a compact support so that they do not belong to X. Therefore, we modify them through multiplication with a smooth window function H. Let $H : \mathbb{R} \to \mathbb{R}$ be an even function, arbitrary often differentiable with compact support in $[-\Omega, \Omega]$, so that for all $u \in [-\Omega_0, \Omega_0] \subset [-\Omega, \Omega]$ there is $H(u) = 1$. Function H can be chosen such that $|H(u)| \leq 1$, $u \in \mathbb{R}$. Now we define the test elements

$$
h_n(u) := \tilde{h}_n(u)H(u).
$$

Obviously, $\|h_n\|_X = \|h_n\|_{r,\infty} \leq C_1$, giving (6.2). Also, $S_\delta h_n \leq 4\|h_n^{(r)}\|_\infty = \mathcal{O}(1)$ and

$$
\begin{aligned}
S_\delta h_n &\leq C_1 \delta^2 \|h_n^{(r+2)}\|_\infty \leq C_1 \delta^2 \frac{1}{n^r} \sum_{k=0}^{r+2} \binom{r+2}{k} n^k \omega_0^k \|H^{(r+2-k)}\|_\infty \\
&\leq C_2 \delta^2 n^2 = C_2 \frac{\sigma(\delta)}{\varphi_n}.
\end{aligned}
$$

That validates (6.3).

For $u \in [-\Omega_1, \Omega_1]$ there is $1 - H(u) = 0$ and

$$
\begin{aligned}
T_{n,u}h_n &\geq n^r \left| \tilde{h}_n(u) - [\tilde{h}_n * n\chi(\cdot n)](u) \right| - \\
&\quad - n^r \left| \tilde{h}_n(u)[1 - H(u)] - [\tilde{h}_n[1 - H] * n\chi(\cdot n)](u) \right| \\
&= |1 - \chi^\wedge(\omega_0)| - n^r |\tilde{h}_n[1 - H] * n\chi(\cdot n)](u)| \\
&\geq \frac{1}{2} - \left| e^{in\omega_0 u} \int_{-\infty}^{\infty} e^{-in\omega_0 v}[1 - H(u - v)]n\chi(nv)\, dv \right| \\
&= \frac{1}{2} - \left| \int_{\mathbb{R}\setminus[n(u-\Omega_0),n(u+\Omega_0)]} e^{-i\omega_0 v}\left[1 - H\left(u - \frac{v}{n}\right)\right]\chi(v)\, dv \right| \\
&\geq \frac{1}{2} - \int_{\mathbb{R}\setminus[n(u-\Omega_0),n(u+\Omega_0)]} |\chi(v)|\, dv \geq \frac{1}{2} - \int_{\mathbb{R}\setminus[n(\Omega_1-\Omega_0),n(-\Omega_1+\Omega_0)]} |\chi(v)|\, dv.
\end{aligned}
$$

Since $\chi \in L^1(\mathbb{R})$, the last integral converges to zero as $n \to \infty$ (independently of u). We select n_0 such that for all $n > n_0$ and $u \in [-\Omega_1, \Omega_1]$

$$
T_{n,\omega}h_n \geq \frac{1}{4}.
$$

For $u := u_1$ we have shown (6.2).

Now we can apply Theorem 6.1 to get a continuous counterexample h_α with compact support in $[-\Omega, \Omega]$ and

$$
\omega_2(h_\alpha^{(r)}, \delta, C(\mathbb{R})) \leq C\delta^\alpha
$$

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but
$$|h_\alpha(u_1) - (h_\alpha * R\chi(R\cdot))(u_1)| \neq o(R^{-r-\alpha}).$$

By inverse Fourier transform we find the required continuous counterexample $f_\alpha(t) := h_\alpha^\vee(t) := \frac{1}{2\pi} h_\alpha^\wedge(-t)$. We show that $f_\alpha(t)$ belongs to $L^1(\mathbb{R})$. To this end Riemann-Lebesgue-lemma (cf. [22]) gives

$$|f_\alpha(t)| \leq C\omega_{r+2}\left(\frac{\pi}{|t|}, h_\alpha, L^1(\mathbb{R})\right).$$

Obviously $(t \to \pm\infty)$

$$\begin{aligned}
\omega_{r+2}\left(\frac{\pi}{|t|}, h_\alpha, L^1(\mathbb{R})\right) &\leq \left[2\Omega + r\frac{\pi}{|t|}\right]\omega_{r+2}\left(\frac{\pi}{|t|}, h_\alpha, C(\mathbb{R})\right) \\
&\leq C\omega_2\left(\frac{\pi}{|t|}, h_\alpha^{(r)}, C(\mathbb{R})\right) = \mathcal{O}(|t|^{-r-\alpha}).
\end{aligned}$$

Because of $r + \alpha > 1$ the L^1-norm of f_α is finite. f_α can be Fourier transformed and the transform is h_α.

Please note that all test elements h_n are complex-valued and that h_α is complex-valued, too. Real parts $\mathrm{Re}(h_n)$ are even and imaginary parts $\mathrm{Im}(h_n)$ are odd. Because of (6.5) this leads to a counterexample h_α in the frequency domain that has an even real part and an odd imaginary part. Inverse Fourier transform then gives a real-valued counterexample f_α in the time domain. We have shown (1.8).

Getting a real-valued counterexample through inverse Fourier transform is very much simpler than constructing a real-valued counterexample for convolution processes in the frequency domain (cf. [18]).

It remains to prove simultaneous sharpness on $\mathbb{B}$ using condition (W5) in connection with (2.4). To this end let $1 \leq j < n$:

$$\begin{aligned}
T_{n,u}\tilde{h}_j &= n^r|\tilde{h}_j(u) - [\tilde{h}_j * n\chi(\cdot n)](u)| \\
&= \left(\frac{n}{j}\right)^r\left|e^{iu\omega_0 j} - e^{iu\omega_0 j}n\int_{-\infty}^{\infty}e^{-i\omega_0\frac{j}{n}nt}\chi(nt)\,dt\right| \\
&= \left(\frac{n}{j}\right)^r\left|1 - \int_{-\infty}^{\infty}e^{-i\omega_0\frac{j}{n}v}\chi(v)\,dv\right| = \left(\frac{n}{j}\right)^r\left|1 - \chi^\wedge\left(\frac{j}{n}\right)\right| \leq C_j\frac{\left(\frac{n}{j}\right)^r}{n^{r+2}} \leq C_j\varphi_n.
\end{aligned}$$

That enables us to show (6.8) for all $u \in \mathbb{B}$:

$$\begin{aligned}
T_{n,u}h_j &\leq n^r\left[\left|\tilde{h}_j(u) - [\tilde{h}_j * n\chi(\cdot n)](u)\right| + \left|\tilde{h}_j(u)[1 - H(u)] - [\tilde{h}_j[1 - H] * n\chi(\cdot n)](u)\right|\right] \\
&\leq C\varphi_n + n^r|\tilde{h}_j[1 - H] * n\chi(\cdot n)](u)| \\
&= C\varphi_n + \left(\frac{n}{j}\right)^r\left|\int_{\mathbb{R}\setminus[n(u-\Omega_0),n(u+\Omega_0)]}e^{-i\omega_0\frac{j}{n}v}\left[1 - H\left(u - \frac{v}{n}\right)\right]\chi(v)\,dv\right| \\
&\leq C\varphi_n + \left(\frac{n}{j}\right)^r\int_{\mathbb{R}\setminus[n(\Omega_1-\Omega_0),n(-\Omega_1+\Omega_0)]}|\chi(v)|\,dv = C\varphi_n + 2\left(\frac{n}{j}\right)^r\int_{n(\Omega_0-\Omega_1)}^{\infty}|\chi(v)|\,dv.
\end{aligned}$$

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Tail condition (2.1) completes the estimate against φ_n with constants not depending on u or n:

$$
\begin{aligned}
T_{n,u} h_j & \leq C\varphi_n + C_2 \left(\frac{n}{j}\right)^r \frac{-2}{r+2} \left[\frac{1}{u^{r+2}}\right]_{n(\Omega_0 - \Omega_1)}^{\infty} \\
& = C\varphi_n + C_2 \left(\frac{1}{j}\right)^r \frac{2}{r+2} \frac{1}{(\Omega_0 - \Omega_1)^{r+2}} \varphi_n.
\end{aligned}
$$

Please note, that we already have shown (6.7) and (6.9). With respect to (6.6) this brings the proof of Theorem 1.2 to an end. We refer to our previous remark that counterexample f_α (created by inverse Fourier transform of h_α) also is real-valued in the context of simultaneous sharpness because of (6.5).

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